The Equivalence of Panel Data Estimators Under Orthogonal Experimental Design

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Abstract

This paper demonstrates the equivalence between pooled OLS, Fixed Effects, and Random Effects estimates when applied to data generated from an orthogonal experimental design under certain conditions. We show that the point estimates of the treatment effects are identical between these three panel data estimators but that the estimated standard errors differ. Specifically, the estimated variance covariance matrices are identical between FE and RE but differ from that of OLS. Despite the equivalence it is meaningful to test for OLS vs FE/RE because the error distributional assumptions are different.

In the conduct of controlled laboratory experiments the virtues of an orthogonal experimental design are well known. For data analytic purposes in both experimental and nonexperimental settings, the advantages of panel data methods are widely recognized. Because of repeated observations in experiments, experimental data often constitute a panel. The presence of subject heterogeneity can lead to inefficient estimation by pooled OLS. Under these circumstances either fixed effects (FE) or random effects (RE) would be the estimator of choice. In this paper, we show that with a panel data set comprised of orthogonal treatment indicator variables in which every cross-sectional unit faces each treatment an equal number of times, OLS, FE, and RE yield identical treatment effect estimates. Although the standard errors are identical for FE and RE, they differ from the conventional Classical Regression Model (CRM) standard errors. In a somewhat different context Oaxaca and Geisler (2003) demonstrate the equivalence between pooled OLS estimates of the effects of time-invariant regressors and a two-stage (feasible) GLS estimator of these effects. The importance of our results for researchers, especially experimental economists, is two-fold. First, we show that the choice between a fixed and random effects estimator is moot in the present context, because these are the same estimator. Hence, the need to decide whether to condition or not on the subject sample does not arise. Secondly, we show that the only remaining choice is to decide whether to use the pooled OLS standard errors or the FE/RE standard errors, which can be accomplished with a standard F-test.

Proof

We begin with a general specification of a balanced design experimental treatment model:

$$Y_{it} = \alpha + X_{it}\beta + \epsilon_{it}$$

where X_{it} is a 1xK vector of treatment indicator variables, β is a Kx1 vector of treatment effects, $i = 1, ..., N$ (subjects) and $t = 1, ..., T$. Since one treatment indicator variable is left out for the reference group, there are a total of K+1 treatments. Without loss of generality we designate treatment 1 as the omitted reference group. Given that the treatments are exogenously assigned, there is no correlation between X_{it} and the disturbance term ϵ_{it}. In the case of the CRM pooled OLS would be the estimator of choice since ϵ_{it} is i.i.d. and satisfies all of the classical assumptions. The FE model arises if the intercept terms α_i vary across subjects:

$$Y_{it} = \alpha_i + X_{it}\beta + \epsilon_{it}.$$

The FE model is efficiently estimated by pooled OLS with subject indicator variables (LSDV) or equivalently in group deviation form (the within estimator). Finally, the RE model arises if we assume that $\alpha_i = \alpha + u_i$:

$$Y_{it} = \alpha + X_{it}\beta + \epsilon_{it} + u_i,$$

where u_i is assumed to be i.i.d. and by the experimental design would be uncorelated with X_{it}. Since the error process in the RE model is associated with a block diagonal disturbance variance/covariance matrix, the model is efficiently estimated by GLS (or FGLS). The appropriate treatment effect estimators corresponding to the CRM, FE, and RE models are given by

$$\begin{aligned}
\hat{\beta}^{\text{ols}} &= \left[X'\left(\mathrm{I}_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)X\right]^{-1} X'\left(\mathrm{I}_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)Y \\
\hat{\beta}^{\text{fe}} &= \left\{X'\left[\mathrm{I}_{NT} - \left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]X\right\}^{-1} X'\left[\mathrm{I}_{NT} - \left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]Y \\
\hat{\beta}^{\text{re}} &= \sigma_\epsilon^2\left\{\psi[X'\left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)X - X'\left(\frac{\iota_{NT}\iota'_{NT}}{NT}\right)X] + X'\left[\mathrm{I}_{NT} - \left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]X\right\}^{-1} \\
&\quad \cdot\frac{1}{\sigma_\epsilon^2}\left\{\psi\left[X'\left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)Y - X'\left(\frac{\iota_{NT}\iota'_{NT}}{NT}\right)\right]Y + X'\left[\mathrm{I}_{NT} - \left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]Y\right\}
\end{aligned}$$

where X is a NTxK observation matrix on the treatment indicator variables, Y is a NTx1 vector of observations on the experimental outcome variable, ι_{NT} and ι_T are NTx1 and Tx1 vectors of 1's, and $\psi = \dfrac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + T\sigma_u^2}$.[1]

Let p equal the number of rounds each treatment is administered. Then each treatment will appear pN times in the sample and $T = p(K+1)$ is the number of observations per subject. The orthogonality of the experimental design for treatment

[1] See Judge, et. al (1980, p. 332) for the specification of the RE estimator of the slope coefficients.

effects yields the following cross-product matrix for treatments:

$$X'X = \begin{bmatrix} pN & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & pN & 0 & \cdot & \cdot & \cdot & 0 \\ & & \cdot & & & & \\ & & \cdot & & & & \\ & & \cdot & & & & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & pN \end{bmatrix}.$$

We will first establish the equivalence between pooled OLS and fixed effects by showing that the terms $X'\left(I_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)X$ and $X'\left(I_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)Y$ in the expression for $\hat{\beta}^{\text{ols}}$ are identical, respectively, to the terms $X'\left[I_{NT} - \left(I_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]X$ and $X'\left[I_{NT} - \left(I_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]Y$ in the expression for $\hat{\beta}^{\text{fe}}$. For pooled OLS estimation of the treatment effects, the cross-product matrices are expressed in deviation form:

$$\begin{aligned} X'\left(\mathrm{I}_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)X &= \begin{bmatrix} pN(1-\frac{1}{K+1}) & -\frac{pN}{K+1} & -\frac{pN}{K+1} & \cdot & \cdot & \cdot & -\frac{pN}{K+1} \\ -\frac{pN}{K+1} & pN(1-\frac{1}{K+1}) & -\frac{pN}{K+1} & \cdot & \cdot & \cdot & -\frac{pN}{K+1} \\ & & \cdot & & & & \\ & & \cdot & & & & \\ & & \cdot & & & & \\ -\frac{pN}{K+1} & -\frac{pN}{K+1} & -\frac{pN}{K+1} & \cdot & \cdot & \cdot & pN(1-\frac{1}{K+1}) \end{bmatrix} \\ &= \frac{pN}{K+1}\begin{bmatrix} K & -1 & -1 & \cdot & \cdot & \cdot & -1 \\ -1 & K & -1 & \cdot & \cdot & \cdot & -1 \\ & & \cdot & & & & \\ & & \cdot & & & & \\ & & \cdot & & & & \\ -1 & -1 & -1 & \cdot & \cdot & \cdot & K \end{bmatrix} \end{aligned}$$

and

$$X'\left(\mathrm{I}_{NT}-\frac{\iota_{NT}\iota'_{NT}}{NT}\right)Y = \begin{pmatrix} \sum_i\sum_t T_{2it}(Y_{it}-\bar{Y}) \\ \sum_i\sum_t T_{3it}(Y_{it}-\bar{Y}) \\ . \\ . \\ . \\ \sum_i\sum_t T_{K+1it}(Y_{it}-\bar{Y}) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i\sum_t T_{2it}Y_{it}-pN\bar{Y} \\ \sum_i\sum_t T_{3it}Y_{it}-pN\bar{Y} \\ . \\ . \\ . \\ \sum_i\sum_t T_{K+1it}Y_{it}-pN\bar{Y} \end{pmatrix},$$

where T_{kit} is an indicator for the kth treatment and $\bar{Y}$ is the pooled sample mean value of Y_{it}.

In the case of fixed effects, the variables are expressed in group deviation form. A typical diagonal element of the FE cross-product matrix $X'\left[I_{NT}-\left(I_N\otimes\frac{\iota_T\iota'_T}{T}\right)\right]X$ may be expressed as $\sum_i\sum_t(T_{kit}-\bar{T}_{ki})^2$ where $\bar{T}_{ki}$ is the mean of treatment variable T_{kit} for the ith subject. This mean is easily seen to be $\frac{p}{p(K+1)} = \frac{1}{K+1}$. Note that $(T_{kit})^2 = T_{kit}$ because the treatment variables are indicator variables. Therefore,

$$\begin{aligned}\sum_i\sum_t(T_{kit}-\bar{T}_{ki})^2 &= \sum_i\sum_t\left(T_{kit}-\frac{1}{K+1}\right)^2 \\ &= \sum_i\sum_t\left[(T_{kit})-\frac{2}{K+1}T_{kit}+\frac{1}{(K+1)^2}\right] \\ &= \sum_i\sum_t T_{kit}-\frac{2}{K+1}\sum_i\sum_t T_{kit}+\sum_i\sum_t\frac{1}{(K+1)^2}\end{aligned}$$

$$
\begin{aligned}
&= pN - \frac{2pN}{K+1} + \frac{pN(K+1)}{(K+1)^2} \\
&= \frac{pNK}{K+1},
\end{aligned}
$$

which is equal to the diagonal elements of the OLS cross-product matrix $X'\left(I_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)X$. A typical off diagonal element of the FE cross-product matrix $X'\left[I_{NT} - \left(I_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]X$ is given by $\sum_i\sum_t(T_{kit} - \bar{T}_{ki})(T_{sit} - \bar{T}_{sit})$ for $k \neq s$. Therefore,

$$
\begin{aligned}
\sum_i\sum_t(T_{kit} - \bar{T}_{ki})(T_{sit} - \bar{T}_{sit}) &= \sum_i\sum_t(T_{kit} - \frac{1}{K+1})(T_{sit} - \frac{1}{K+1}) \\
&= \sum_i\sum_t(T_{kit}T_{sit} - \frac{T_{kit}}{K+1} - \frac{T_{sit}}{K+1} + \frac{1}{(K+1)^2}) \\
&= \sum_i\sum_t T_{kit}T_{sit} - \frac{1}{K+1}\sum_i\sum_t T_{kit} \\
&\quad - \frac{1}{K+1}\sum_i\sum_t T_{sit} + \sum_i\sum_t\frac{1}{(K+1)^2} \\
&= 0 - \frac{pN}{K+1} - \frac{pN}{K+1} + \frac{pN(K+1)}{(K+1)^2} \\
&= -\frac{pN}{K+1}
\end{aligned}
$$

which is equal to the off diagonal elements of the OLS cross-product matrix $X'\left(I_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)X$. Therefore, $X'\left[\mathrm{I}_{NT} - \left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]X = X'\left(I_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)X$.

Next we consider the elements of the FE cross-product vector $X'\left[I_{NT} - \left(I_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]Y$. A typical element of this vector would be expressed as $\sum_i\sum_t(T_{kit} - \bar{T}_{ki})(Y_{it} - \bar{Y}_i)$ where $\bar{Y}_i$ is the mean of experimental outcome variable Y_{it} for the ith subject. Accordingly,

$$
\begin{aligned}
\sum_i\sum_t(T_{kit} - \bar{T}_{ki})(Y_{it} - \bar{Y}_i) &= \sum_i\sum_t\left(T_{kit} - \frac{1}{K+1}\right)(Y_{it} - \bar{Y}_i) \\
&= \sum_i\sum_t\left(T_{kit}Y_{it} - T_{kit}\bar{Y}_i - \frac{Y_{it}}{K+1}\right. \\
&\quad \left. + \frac{\bar{Y}_i}{K+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_i \sum_t T_{kit} Y_{it} - \sum_i \bar{Y}_i \sum_t T_{kit} \\
&\quad - \frac{1}{K+1} \sum_i \sum_t Y_{it} + \frac{1}{K+1} \sum_i \sum_t \bar{Y}_i \\
&= \sum_i \sum_t T_{kit} Y_{it} - p \sum_i \bar{Y}_i - pN\bar{Y} \\
&\quad + \frac{p(K+1)}{K+1} \sum_i \bar{Y}_i \\
&= \sum_i \sum_t T_{kit} Y_{it} - pN\bar{Y},
\end{aligned}
$$

which is equal to the kth element of the OLS cross-product vector $X' \left(I_{NT} - \frac{\iota_{NT} \iota'_{NT}}{NT} \right) Y$. Thus, $X' \left[I_{NT} - \left(I_N \otimes \frac{\iota_T \iota'_T}{T} \right) \right] Y = X' \left(I_{NT} - \frac{\iota_{NT} \iota'_{NT}}{NT} \right) Y$. This establishes the equivalence between the OLS and FE estimates of the treatment effects.

It remains to establish the equivalence between the FE and RE estimates of the treatment effects. We will show that in the estimator formula for β^{re} it is the case that

$$
X' \left(I_N \otimes \frac{\iota_T \iota'_T}{T} \right) X - X' \left(\frac{\iota_{NT} \iota'_{NT}}{NT} \right) X = \underset{K\mathrm{x}K}{0}
$$

and

$$
X' \left(I_N \otimes \frac{\iota_T \iota'_T}{T} \right) Y - X' \left(\frac{\iota_{NT} \iota'_{NT}}{NT} \right) Y = \underset{K\mathrm{x}1}{0}.
$$

These conditions imply that $\beta^{re} = \beta^{fe}$. First consider the term $X' \left(\frac{\iota_{NT} \iota'_{NT}}{NT} \right) X$. Let $X = (T_2, ...T_{K+1})$ where each T_k column of X is a NTx1 vector of observations on the indicator for the kth treatment. It follows that

$$
X' \left(\frac{\iota_{NT} \iota'_{NT}}{NT} \right) X = \frac{1}{NT} (T_2, ...T_{K+1})' \iota_{NT} \iota'_{NT} (T_2, ...T_{K+1})
$$

$$
= \frac{1}{NT} \underset{(K\text{x}NT)}{\begin{bmatrix} Np & \cdot & \cdot & \cdot & Np \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ Np & \cdot & \cdot & \cdot & Np \end{bmatrix}} (T_2, ...T_{K+1})
$$

$$
= \frac{1}{NT} \underset{(K\text{x}K)}{\begin{bmatrix} (Np)^2 & \cdot & \cdot & \cdot & (Np)^2 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ (Np)^2 & \cdot & \cdot & \cdot & (Np)^2 \end{bmatrix}}
$$

$$
= \underset{(K\text{x}K)}{\begin{bmatrix} \frac{Np}{K+1} & \cdot & \cdot & \cdot & \frac{Np}{K+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \frac{Np}{K+1} & \cdot & \cdot & \cdot & \frac{Np}{K+1} \end{bmatrix}}
$$

since $NT = Np(K+1)$.

The term $X'\left(I_N \otimes \frac{\iota_T \iota_T'}{T}\right) X$ that appears in the estimation formula for $\hat{\beta}^{re}$ will be shown to be identical to $X'\left(\frac{\iota_{NT}\iota_{NT}'}{NT}\right) X$. Note first that $\mathrm{I}_N \otimes \frac{\iota_T \iota_T'}{T}$ is a NTxNT block diagonal matrix in which the diagonal blocks are TxT matrices whose elements are simply $1/T$. All other elements of $\mathrm{I}_N \otimes \frac{\iota_T \iota_T'}{T}$ are 0. It is easily seen that

$$X'\left(I_N \otimes \frac{\iota_T \iota_T'}{T}\right) = \frac{1}{T} \underset{K\text{x}NT}{\begin{bmatrix} p & \cdot & \cdot & \cdot & p \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ p & \cdot & \cdot & \cdot & p \end{bmatrix}}.$$

Post multiplying $X'\left(I_N \otimes \frac{\iota_T \iota_T'}{T}\right)$ by X yields

$$\begin{aligned}
X'\left(I_N \otimes \frac{\iota_T \iota_T'}{T}\right) X &= \frac{1}{T} \underset{K\text{x}NT}{\begin{bmatrix} p & \cdot & \cdot & \cdot & p \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ p & \cdot & \cdot & \cdot & p \end{bmatrix}} \underset{NT\text{x}K}{X} \\
&= \frac{1}{T} \underset{K\text{x}K}{\begin{bmatrix} Np^2 & \cdot & \cdot & \cdot & Np^2 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ Np^2 & \cdot & \cdot & \cdot & Np^2 \end{bmatrix}} \\
&= \underset{(K\text{x}K)}{\begin{bmatrix} \frac{Np}{K+1} & \cdot & \cdot & \cdot & \frac{Np}{K+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \frac{Np}{K+1} & \cdot & \cdot & \cdot & \frac{Np}{K+1} \end{bmatrix}} \text{ since } T = p(K+1) \\
&= X'\left(\frac{\iota_{NT} \iota_{NT}'}{NT}\right) X.
\end{aligned}$$

Finally it remains to be shown that $X'\left(I_N \otimes \frac{\iota_T \iota_T'}{T}\right) Y = X'\left(\frac{\iota_{NT} \iota_{NT}'}{NT}\right) Y$. We will first consider $X'\left(\frac{\iota_{NT}\iota'_{NT}}{NT}\right)Y$:

$$X'\left(\frac{\iota_{NT}\iota'_{NT}}{NT}\right)Y = \frac{1}{NT}\underset{(K\text{x}NT)}{\begin{bmatrix} Np & \cdot & \cdot & \cdot & Np \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ Np & \cdot & \cdot & \cdot & Np \end{bmatrix}}\underset{NT\text{x}1}{Y}$$

$$= \frac{1}{NT}\underset{K\text{x}1}{\begin{bmatrix} Np\sum_i\sum_t Y_{it} \\ \cdot \\ \cdot \\ \cdot \\ Np\sum_i\sum_t Y_{it} \end{bmatrix}}$$

$$= Np\bar{Y}\ \iota_K$$

where i_K is a Kx1 vector of 1's. Now consider the term $X'\left(I_N \otimes \frac{\iota_T\iota'_T}{T}\right)Y$:

$$X'\left(I_N \otimes \frac{\iota_T\iota'_T}{T}\right)Y = \frac{1}{T}\underset{K\text{x}NT}{\begin{bmatrix} p & \cdot & \cdot & \cdot & p \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ p & \cdot & \cdot & \cdot & p \end{bmatrix}}\underset{NT\text{x}1}{Y}$$

$$= \frac{1}{T}\underset{K\text{x}1}{\begin{bmatrix} p\sum_i\sum_t Y_{it} \\ \cdot \\ \cdot \\ \cdot \\ p\sum_i\sum_t Y_{it} \end{bmatrix}}$$

$$
\begin{aligned}
&= \frac{1}{NT}\begin{bmatrix} Np\sum_i\sum_t Y_{it} \\ . \\ . \\ . \\ Np\sum_i\sum_t Y_{it} \end{bmatrix}_{K\text{x}1} \text{ upon multiplying and dividing by } N \\
&= Np\bar{Y}\ \iota_K \\
&= X'\left(\frac{\iota_{NT}\iota'_{NT}}{NT}\right)Y.
\end{aligned}
$$

This establishes the equivalence of random effects and fixed effects estimation and hence the equivalence between random effects and pooled OLS. In the case of random effects, specification of the transformed model in observation form yields variables of the type $Y_{it} - \theta\bar{Y}_i$ where $\theta = 1 - \psi^{\frac{1}{2}}$. Consequently, any arbitrarily chosen value of ψ will lead to the same coefficient estimates because ψ is multiplied by a null matrix. However, the variance covariance matrix is not invariant with respect to the choice of ψ if the variance of ε were arbitrarily chosen.

The estimated variance covariance matrices will be identical between fixed and random effects but different from pooled OLS:

$$
\begin{aligned}
\widehat{Var}\left(\hat{\beta}^{ols}\right) &= \hat{\sigma}^2_{\varepsilon_{\text{ols}}}\left[X'\left(\mathrm{I}_{NT} - \frac{\iota_{NT}\iota'_{NT}}{NT}\right)X\right]^{-1} \\
\widehat{Var}\left(\hat{\beta}^{re}\right) &= \widehat{Var}\left(\hat{\beta}^{fe}\right) \\
&= \hat{\sigma}^2_{\varepsilon_{\text{fe}}}\left\{X'\left[\mathrm{I}_{NT} - \left(\mathrm{I}_N \otimes \frac{\iota_T\iota'_T}{T}\right)\right]X\right\}^{-1}
\end{aligned}
$$

where $\hat{\sigma}^2_{\varepsilon_{\text{ols}}} = \dfrac{\hat{\varepsilon}'_{\text{ols}}\hat{\varepsilon}_{\text{ols}}}{NT - (K+1)}$ and $\hat{\sigma}^2_{\varepsilon_{\text{fe}}} = \dfrac{\hat{\varepsilon}'_{\text{fe}}\hat{\varepsilon}_{\text{fe}}}{NT - (N+K)}$. The question of which set of standard errors are appropriate to use is a question of which estimator/model is appropriate. Even though the point estimates of the treatment effects are identical between CRM and FE/RE, the distributional assumptions about the error terms are not the same. The standard F test for CRM vs FE can be used to determine which model's distributional assumptions are appropriate. To see this, we will examine the relationship between the OLS residuals and the FE residuals. Note

$$\begin{aligned}
\hat{\varepsilon}_{it}^{\text{ols}} - \hat{\varepsilon}_{it}^{\text{fe}} &= \left[(Y_{it} - \bar{Y}) - (X_{it} - \bar{X})\hat{\beta}\right] - \left[(Y_{it} - \bar{Y}_i) - (X_{it} - \bar{X}_i)\hat{\beta}\right] \\
&= \left(\bar{Y}_i - \bar{X}_i\hat{\beta}\right) - \left(\bar{Y} - \bar{X}\hat{\beta}\right) \\
&= \hat{\alpha}_i - \hat{\alpha}
\end{aligned}$$

where $\hat{\beta} = \hat{\beta}^{\text{ols}} = \hat{\beta}^{\text{fe}} = \hat{\beta}^{\text{re}}$. Since $\sum_i \sum_t \hat{\varepsilon}_{it}^{\text{ols}} - \sum_i \sum_t \hat{\varepsilon}_{it}^{\text{fe}} = 0$, it is easily seen that $\sum_i \sum_t (\hat{\alpha}_i - \hat{\alpha}) = 0 \Rightarrow \sum_i [p(K+1)](\hat{\alpha}_i - \hat{\alpha}) = 0 \Rightarrow \hat{\alpha} = \frac{\sum_i \hat{\alpha}_i}{N}$. Thus, the estimated constant term in the CRS model is the average of the estimated individual fixed effects. We can express the OLS residuals in terms of the FE residuals: $\hat{\varepsilon}_{it}^{\text{ols}} = \hat{\varepsilon}_{it}^{\text{fe}} + \hat{\alpha}_i - \hat{\alpha}$. Now squaring both sides of the preceding expression and summing over the t index yields:

$$\begin{aligned}
\sum_t \left(\hat{\varepsilon}_{it}^{\text{ols}}\right)^2 &= \sum_t \left(\hat{\varepsilon}_{it}^{\text{fe}}\right)^2 + \sum_t (\hat{\alpha}_i - \hat{\alpha})^2 + 2(\hat{\alpha}_i - \hat{\alpha}) \sum_t \hat{\varepsilon}_{it}^{\text{fe}} \\
&= \sum_t \left(\hat{\varepsilon}_{it}^{\text{fe}}\right)^2 + p(K+1)(\hat{\alpha}_i - \hat{\alpha})^2 .
\end{aligned}$$

Next, we sum over the index i to obtain

$$\sum_i \sum_t \left(\hat{\varepsilon}_{it}^{\text{ols}}\right)^2 = \sum_i \sum_t \left(\hat{\varepsilon}_{it}^{\text{fe}}\right)^2 + p(K+1) \sum_i (\hat{\alpha}_i - \hat{\alpha})^2$$

or in vector notation

$$\hat{\varepsilon}_{\text{ols}}' \hat{\varepsilon}_{\text{ols}} = \hat{\varepsilon}_{\text{fe}}' \hat{\varepsilon}_{\text{fe}} + p(K+1) \sum_i (\hat{\alpha}_i - \hat{\alpha})^2 .$$

This leads to the familiar result that the F test of CRM vs FE based on the difference in restricted and unrestricted residuals, $\hat{\varepsilon}_{\text{ols}}' \hat{\varepsilon}_{\text{ols}} - \hat{\varepsilon}_{\text{fe}}' \hat{\varepsilon}_{\text{fe}} = p(K+1) \sum_i (\hat{\alpha}_i - \hat{\alpha})^2$, is a test of the equality of the fixed effects. Rejection of OLS would suggest that the FE standard errors are the appropriate ones.

An interesting result arises with respect to estimating the variance of the random effects in the context of our orthogonal design. Typically, one estimates the between or group means model by OLS to obtain an estimate of σ_u^2:

$$\bar{Y}_i = \alpha + \bar{X}_i\beta + \omega_i,\ i = 1, ..., N$$

where $\omega_i = u_i + \bar{\varepsilon}_i$. Since $\sigma_\omega^2 = \sigma_u^2 + \dfrac{\sigma_\varepsilon^2}{T}$, one obtains $\sigma_u^2 = \sigma_\omega^2 - \dfrac{\sigma_\varepsilon^2}{T}$ as a residual. In the present context the group means model cannot be estimated because the values of the variables comprising $\bar{X}_i$ are identically equal to $\frac{1}{K+1}$ and therefore perfect multicollinearity is present. However, it is still possible to consistently estimate σ_u^2. The estimated random effects specification of the model can be expressed as

$$\begin{aligned} Y_{it} &= \hat{\alpha} + X_{it}\hat{\beta} + \hat{\epsilon}_{it}^{\text{fe}} + \hat{u}_i \text{ which implies} \\ \hat{u}_i &= Y_{it} - \left(\hat{\alpha} + X_{it}\hat{\beta}\right) - \hat{\epsilon}_{it}^{\text{fe}} \\ &= \hat{\epsilon}_{it}^{\text{ols}} - \hat{\epsilon}_{it}^{\text{fe}} \\ &= \hat{\alpha}_i - \hat{\alpha}. \end{aligned}$$

An obvious estimator of σ_u^2 is given by

$$\tilde{\sigma}_u^2 = \frac{\sum_i (\hat{\alpha}_i - \hat{\alpha})^2}{N}.$$

Furthermore, the Breusch-Pagan LM test for CRM vs RE is based exclusively on the pooled OLS residuals from the CRM model. An LM test based on the OLS residuals could also be used to test for CRM vs FE. Although these two LM test statistics are different asymptotically because the degrees of freedom for the former are 1 and for the latter are $N - 1$, they should lead to the same test outcome asymptotically.

The results showing the equivalence between the point estimates of the CRM and FE/RE model generalize to the addition of any set of regressors that are uncorrelated with the treatment variables. In this case, the estimated treatment effects would remain unchanged but, in general, the standard errors will differ from the case without the additional regressors. However, once the additional regressors are added, the new standard errors are the same between fixed and random effects, as before.

Summary and Conclusions

We show in this paper that, in certain contexts, the CRM, FE, and RE models will yield identical experimental treatment effect estimates. Furthermore, the standard errors for FE and RE are identical but differ from those of the CRM. This implies equivalence of the FE and RE estimators in these contexts. The experimentalist is therefore relieved of the need to test for random versus fixed effects. However, the experimentalist would still need to select the appropriate standard errors from among the CRM or FE/RE models. This is accomplished with a straightforward F-test as we show in the paper.

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